J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 29–42
Printed in Great Britain

Finite amplitude convective cells and continental drift

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(Received 14 July 1966)

A solution is obtained for steady, cellular convection when the Rayleigh number and the Prandtl number are large. The core of each two-dimensional cell contains a highly viscous, isothermal flow. Adjacent to the horizontal boundaries are thin thermal boundary layers. On the vertical boundaries between cells thin thermal plumes drive the viscous flow. The non-dimensional velocities and heat transfer between the horizontal boundaries are found to be functions only of the Rayleigh number. The theory is used to test the hypothesis of large scale convective cells in the earth's mantle. Using accepted values of the Rayleigh number for the earth's mantle the theory predicts the generally accepted velocity associated with continental drift. The theory also predicts values for the heat flux to the earth's surface which are in good agreement with measurements carried out on the ocean floors.

1. Introduction

This investigation concerns the steady cellular convection which occurs when a layer of fluid in a force field is heated from below. The analysis will be restricted to the two-dimensional problem with free-surface boundary conditions. The solution of the linearized stability problem has been given by Rayleigh (1916). He found that there was an onset of convective motion when the non-dimensional parameter $R = \alpha(T_{w1} - T_{w2})d^3g/\kappa\nu$ exceeds a critical value, where α is the coefficient of thermal expansion, g the acceleration of gravity, κ the thermal diffusivity, ν the kinematic viscosity, d the thickness of the layer of fluid, T_{w1} the temperature of the hot lower surface, and T_{w2} the temperature of the cold upper surface. This parameter is now known as the Rayleigh number and for free-surface boundary conditions the critical Rayleigh number for the onset of convection is 657. The wavelength of the convective motion is $\lambda = 2^{\frac{3}{2}}d$.

When the Rayleigh number is large compared with the critical Rayleigh number the linear theory is no longer valid and it is necessary to solve a set of nonlinear partial differential equations. Extensions of the linear theory into the nonlinear régime have been given by Malkus & Veronis (1958), Kuo (1961) and Platzman (1965). However, these theories are expansions about the critical

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Rayleigh number and are not expected to be valid when the Rayleigh number is large compared with the critical Rayleigh number. Numerical computations of cellular convection for Prandtl numbers of order unity have been given by Fromm (1965).

In non-linear convection the Prandtl number, $Pr = \nu/\kappa$, enters the solution as well as the Rayleigh number. It has been pointed out by Kuo (1961) that for large Rayleigh numbers and large Prandtl numbers an isothermal core develops in each convective cell. Similar results have been obtained for natural convection in closed cavities by Batchelor (1954) and Weinbaum (1964). The analysis given in this paper is concerned with this limit, i.e. the Rayleigh number is large compared with the critical Rayleigh number and the Prandtl number is large compared with one. A model of the convective motion is used in which the core of each cell is highly viscous and isothermal. Adjacent to the horizontal boundaries are thin thermal boundary layers and along the vertical boundaries between cells are thin thermal plumes. When the Prandtl number is large the onset of unsteady, turbulent convection is inhibited.

One of the most interesting applications of cellular convection is in the earth's mantle. The hypothesis of large scale cellular convection is favoured as the driving mechanism of continental drift. Numerous workers have postulated the existence of such currents both on theoretical grounds and in order to explain a variety of thermal and kinetic phenomena at the earth's surface. Although the earth's mantle is composed of crystalline ferromagnesium silicates, over long periods of time and at high temperatures the mantle is expected to display rheid behaviour.

Convective currents at present operating in the mantle are thought to have the form of elongated rolls, tens of thousands of kilometres long and several thousand kilometres wide (Girdler 1965). The regions of converging, falling convection may be characterized by deep and intermediate focus earthquakes. Surface features may include island-arc areas, deep sea trenches, and other compressional phenomena. The regions of rising, diverging currents are associated with shallow seismicity. Surface features are the mid-oceanic ridges; e.g. the Mid-Atlantic Ridge and the East Pacific Rise. These ridges form part of a world-wide oceanic ridge system and display extensional features which suggest that the crust is moving away from them on either side. In addition the ridges are characterized by volcanic activity and a surface heat flux which may be as much as six times the average for the ocean floors. Dating of volcanic islands associated with mid-oceanic ridges indicates that they have been convected away from their origin (Wilson 1965). A summary of the geological arguments which favour the convective current hypothesis has been given by Girdler (1965).

Although qualitative comparisons strongly favour the convective hypothesis, agreement with a quantitative theory would greatly strengthen the case. A complete analysis of the problem is prohibitively difficult. The heating of the interior of the earth causes a heat flux to the earth's surface. This occurs in the presence of a radial gravitational field and the rotation of the earth. An essential question is whether the material behaves as a liquid and if it does what are its properties. Many authors have considered these problems. It is generally accepted that the earth's mantle displays fluid behaviour and may be treated as a fluid using an appropriate value of the viscosity. A detailed discussion of the physical properties of the mantle will be given in a later section. It has been shown by Jeffreys (1928) that the influence of the earth's rotation upon convection in the mantle is negligible. Jeffreys (1930) has also shown that the Boussinesq approximation may be used if the temperature in the Boussinesq equation is taken to be the difference between the actual temperature and the adiabatic temperature provided this temperature difference is not too large. Chandrasekhar (1961) has considered the effect of spherical symmetry on the problem. Since it is expected that mantle convection will be restricted to a thin layer it is a good approximation to analyse the equivalent planar problem. Also, because the layer is thin, it is appropriate to neglect the heat produced by radioactivity within the convecting layer. Models for the earth's mantle which postulate a strong concentration of radioactivity near the surface are based on the assumption that heat transfer within the mantle is mainly by conduction and radiation. If the heat transfer is mainly by cellular convection there is no objection to a more uniform and deeper distribution of radioactive heat sources and no reason to assume a near-surface concentration.

An extensive discussion of how the linearized analysis of cellular convection can be applied to convection in the earth's mantle has been given by Knopoff (1964). In his analysis Knopoff takes the Rayleigh number for the earth's mantle to be 10^6–10^8 and the Prandtl number to be 10^{21}–10^{23}. Therefore an analysis valid for large values of the Rayleigh number and Prandtl number should be applicable to the earth's mantle.

2. The steady convection solution

A. Basic equations

Our goal is to solve for the steady convective flow between two horizontal boundaries when a body force acts downwards and the lower boundary is maintained at a higher temperature than the upper boundary. The Rayleigh number is large compared with the critical Rayleigh number and the Prandtl number is large compared with one. The flow is assumed to be laminar. The Boussinesq approximation is used, that is, in writing the conservation equations the density and the coefficients (viscosity, thermal diffusivity, etc.) are assumed to be constant except for the density in the body force term of the momentum equation. A linear relation is assumed between the variations of temperature and density,

$$\rho' - \rho_0' = -\rho_0' \alpha (T' - T_0'), \tag{1}$$

where T_0' is the temperature at which $\rho' = \rho_0'$. Primes denote dimensional varibles. Introducing $\theta' = T' - T_0'$ and $P' = p' + \rho_0' g y'$ the equations for conservation of mass, momentum and energy are given by

$$\nabla' . \mathbf{u}' = 0, \tag{2}$$

$$\partial \mathbf{u}'/\partial t' + (\mathbf{u}' . \nabla')\mathbf{u}' = -(\nabla' P/\rho_0') + \nu \nabla'^2 \mathbf{u}' + \alpha \theta' g \mathbf{j}, \tag{3}$$

$$(\partial \theta'/\partial t') + (\mathbf{u}' . \nabla')\theta' = \kappa \nabla'^2 \theta. \tag{4}$$

In the absence of convective motions the solution is a linear temperature gradient between the horizontal boundaries.

In order to simplify the analysis we introduce the following non-dimensional variables:

$$\nabla = d\nabla', \quad t = t'\kappa/d^2, \quad u = u'd/\kappa, \quad P = P'd^2/\rho_0'\nu\kappa, \quad \theta = \theta'/\beta d,$$

where d is the distance between the horizontal boundaries and β is the linear temperature gradient which would occur in the absence of convective motions. Substitution of these dimensionless variables into equations (2), (3) and (4) gives

$$\nabla . \mathbf{u} = 0, \tag{5}$$

$$Pr^{-1}\{(\partial\mathbf{u}/\partial t) + [\mathbf{u} . \nabla]\mathbf{u}\} = -\nabla P + \nabla^2\mathbf{u} + R\theta\mathbf{j}, \tag{6}$$

$$(\partial\theta/\partial t) + (\mathbf{u} . \nabla)\theta = \nabla^2\theta, \tag{7}$$

where R is the Rayleigh number and Pr the Prandtl number.

Before solutions of (5), (6) and (7) can be obtained, boundary conditions for both the velocity and temperature on the two horizontal boundaries at $y = 0$ and $y = 1$ are required. We will assume that the temperatures at the upper and lower boundaries are constant, $\theta = \frac{1}{2}$ and $\theta = -\frac{1}{2}$. Only free surface boundary conditions on the velocity will be considered, that is the tangential component of the shear stress at the horizontal boundaries is taken to be zero. Since our analysis will be restricted to two-dimensional flows the free surface boundary conditions require that $\partial u/\partial y = 0$ at $y = 0$ and $y = 1$.

B. Steady convection model

For steady convective motions (6) and (7) become

$$Pr^{-1}(\mathbf{u} . \nabla)\mathbf{u} = -\nabla P + \nabla^2\mathbf{u} + R\theta\mathbf{j}, \tag{8}$$

$$(\mathbf{u} . \nabla)\theta = \nabla^2\theta. \tag{9}$$

If the Prandtl number is large compared with one, the convective terms in the momentum equation may be neglected and (8) reduces to

$$0 = -\nabla P + \nabla^2\mathbf{u} + R\theta\mathbf{j}. \tag{10}$$

This is equivalent to a small Reynolds number approximation in forced convective flows.

Although dropping the convection terms in the momentum equation is a considerable simplification, the equations are still highly non-linear so that it is necessary to prescribe a model before a solution can be obtained. The model used in this paper is illustrated in figure 1. The fluid is confined between the planes $y = 0$ and $y = 1$ and is divided into cellular, two-dimensional rolls; alternate rolls flow in the clockwise and counterclockwise directions. The entire flow field is highly viscous. On the hot and cold boundary planes are thin thermal boundary layers. When the two hot boundary layers from adjacent cells meet they separate from the horizontal plane and form a plume which rises to the upper surface. When the hot plume comes into contact with the upper cold surface, a stagnation point thermal boundary layer is formed. As the flow splits and continues along the cool upper surface this stagnation point boundary layer becomes the cold thermal boundary layer which in turn becomes part of the cold, descending plume. Actually there will be a series of thermal layers as the boundary layers continue to convect in a spiral motion. However, our analysis will be restricted to the first

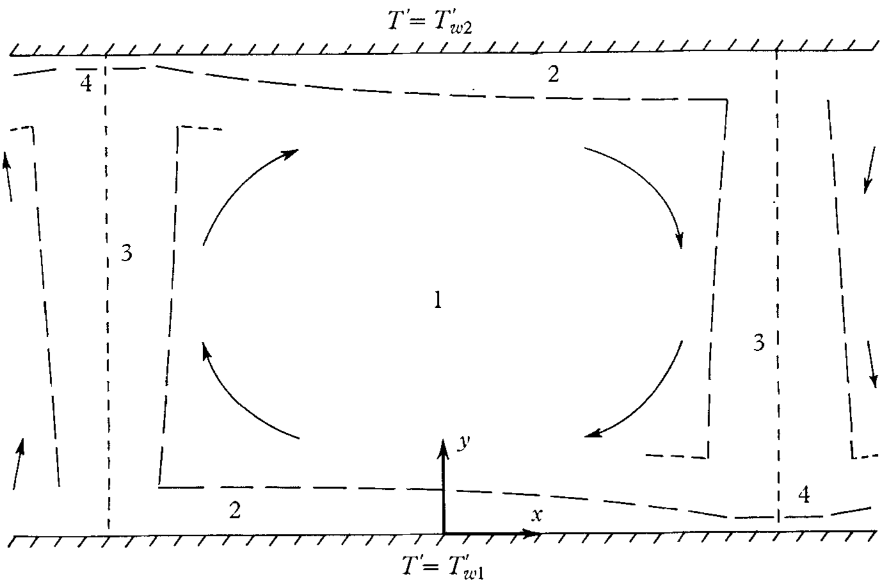

FIGURE 1. Sketch of the steady convection model showing: (1) the isothermal core, (2) thermal boundary layers, (3) thermal convective plumes, and (4) the stagnation point thermal layers.

layer adjacent to the boundaries of the cell. The core will be assumed to be isothermal. This should be a good approximation since most of the temperature drop will occur in the first layer. That the boundary layers and plumes are in fact thin compared with the dimensions of the cell will be verified after the solution is obtained.

Using this model the viscous core is isothermal and has rectangular boundaries since the boundary layers and plumes are thin. The flow in the core is obtained by solving the biharmonic equation in a rectangular region. Since the large viscosity precludes appreciable changes of velocity in the thin thermal layers, it is appropriate to use the velocities obtained from the viscous solution in solving for the temperature profiles in the thermal boundary layers and convective plumes. The magnitude of the velocities is obtained by matching the solutions for the thermal layers to the solution for the core flow.

C. Viscous core

Since the boundary layers and plumes are assumed to be thin, it is appropriate to obtain the two-dimensional core flow in a rectangular cell with dimensions 2δ and 1. Since the core is isothermal the energy equation is not required. The equation for conservation of momentum (10) reduces to

$$-\nabla P + \nabla^2 \mathbf{u} = 0. \tag{11}$$

Introducing the dimensionless stream function ψ, $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, (5) and (11) combine to give the biharmonic equation for the stream function

$$\nabla^4\psi = 0. \tag{12}$$

The biharmonic equation may be used since the body force term in the momentum equation vanishes in the isothermal core and the convection terms may be neglected compared with the viscous terms for large Prandtl numbers. The condition that there be no flow through the boundaries of the cell requires that $u = 0$ at $x = \pm\delta$ and $v = 0$ at $y = 0, 1$. The free-surface boundary conditions require that $\partial u/\partial y = 0$ at $y = 0, 1$. From symmetry it is necessary that $\partial v/\partial x = 0$ at $x = \pm\delta$. However, $\partial v/\partial x$ may vary considerably through the structure of the thin thermal plumes so that it is not appropriate to apply this boundary condition to the core flow. We will take $\partial v/\partial x = \gamma$ at the outer edge of the plume and since the plume is thin it is appropriate to require $\partial v/\partial x = \gamma$ at $x = \pm\delta$ for the core solution. The value of γ will be obtained from the integral of the temperature deficit (excess) in the convective plume. We will show that a constant value of γ is consistent with the model.

By separation of variables it is found that

$$\psi = \sum_n [\sin\alpha_n y(A_n\cosh\alpha_n x + B_n\sinh\alpha_n x + C_n x\cosh\alpha_n x + D_n x\sinh\alpha_n x)] \tag{13}$$

satisfies the biharmonic equation. In order to satisfy the required boundary conditions it is necessary that

$$\alpha_n = \pi n \quad (n = 1, 3, 5, 7, \ldots),$$

$$B_n = C_n = 0, \quad A_n = -\frac{2\gamma\delta\tanh n\pi\delta}{n^2\pi^2\cosh n\pi\delta}, \quad D_n = \frac{2\gamma}{n^2\pi^2\cosh n\pi\delta},$$

and the velocity components within the viscous core are given by

$$u = \sum_{n=1,3,\ldots}\left[\frac{2\gamma\cos n\pi y}{n^2\pi^2\cosh n\pi\delta}(n\pi\delta\tanh n\pi\delta\cosh n\pi x - n\pi x\sinh n\pi x)\right], \tag{14}$$

$$v = \sum_{n=1,3,\ldots}\left[\frac{2\gamma\sin n\pi y}{n^2\pi^2\cosh n\pi\delta}(n\pi x\cosh n\pi x + \{1 - n\pi\delta\tanh n\pi\delta\}\sinh n\pi x)\right]. \tag{15}$$

Before further computations are carried out we will relate the two dimensions of the rectangular cell, 1 and 2δ. We assume that the cell size is the same as that given by the linear theory, namely $\delta = 2^{-\frac{1}{2}}$ since $\delta = \lambda/4d$ and $\lambda = 2^{\frac{3}{2}}d$. The vertical component of the dimensionless velocity at $x = \pm\delta = \pm 2^{-\frac{1}{2}}$ can now be evaluated. The ratio v/γ at $x = \pm 2^{-\frac{1}{2}}$ is plotted against y in figure 2. The mean vertical velocity on the boundary between cells is

$$v = \pm 0{\cdot}141\gamma. \tag{16}$$

Similarly the horizontal component of the dimensionless velocity at $y = 0, 1$, can be evaluated. The ratio u/γ at $y = 0, 1$ is plotted against x in figure 3. The mean horizontal velocity on the horizontal boundaries is

$$u = \pm 0{\cdot}0804\gamma. \tag{17}$$

It is seen that away from the corners the horizontal velocity is nearly constant.

D. Thermal boundary layers

Having obtained the core flow we now turn to the thermal boundary layers on the upper and lower surfaces. Because of the large viscosity (large Prandtl number) the flow velocity will not be a function of y in the thin thermal boundary

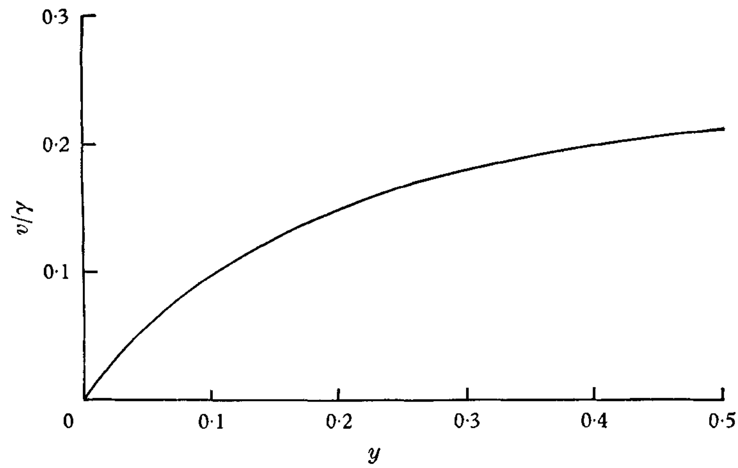

FIGURE 2. Dependence of v/γ at the boundary between cells on y.

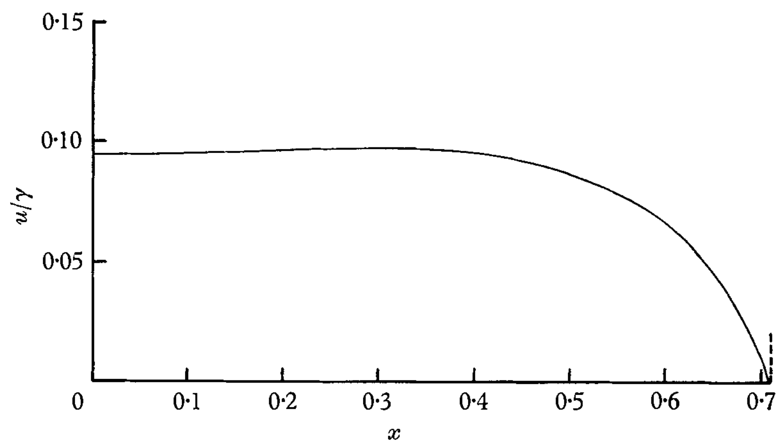

FIGURE 3. Dependence of u/γ at the surface on x.

layers. We will also assume that the velocity is not a function of x and is given by (17). Experience with forced convection boundary layers shows that the constant velocity assumption should not lead to serious errors for the type of velocity distribution given in figure 3. Therefore we assume that u = constant and it follows from the continuity equation that $v = 0$ in the thermal boundary layers.

Having prescribed the velocity, only the energy equation is required to obtain the temperature distribution in the thermal boundary layers. Since each thermal boundary layer is assumed to be thin, it is appropriate to use the boundary-layer form of the energy equation $(\partial^2/\partial y^2) \gg (\partial^2/\partial x^2)$ with the result from (7) that

$$u(\partial\theta/\partial x_1) = \partial^2\theta/\partial y_1^2. \tag{18}$$

For convenience we measure x_1 from the origin of the thermal boundary layer and y_1 is the distance from the surface. From symmetry the core temperature is the mean of the two surface temperatures. Taking the core temperature to be T_0' then $\theta = \theta' = 0$ as $y_1 \to \infty$ in the thermal boundary-layer solution and $\theta = \pm\frac{1}{2}$ at $y_1 = 0$. The analysis is valid for both horizontal surfaces.

A similarity solution of (18) which satisfies these boundary conditions is

$$\theta = \frac{1}{2}\left(1 - \frac{1}{\pi^{\frac{1}{2}}}\int_0^{y_1(u/x_1)^{\frac{1}{2}}} e^{-z^2/4}\,dz\right). \tag{19}$$

This solution is invalid near the origin of the thermal boundary layer where the stagnation point flow must be considered. However, many solutions of boundary-layer problems show that this local failure will not affect the validity of the solution away from the vicinity of the stagnation point. It is also appropriate to measure x_1 from the vertical boundary as long as the stagnation boundary layer is thin compared with the thermal boundary layer on the rest of the surface.

The local heat flux to the surface can be expressed in terms of a local Nusselt number

$$Nu_l = \frac{dq_w'}{k(T_{w1}' - T_{w2}')},$$

where q_w' is the local heat flux per unit area and k is the thermal conductivity. The local Nusselt number obtained from (19) is

$$Nu_l = \tfrac{1}{2}(u/\pi x_1)^{\frac{1}{2}}. \tag{20}$$

The thickness of the thermal boundary layer, $y_{1\delta}$, is defined as the distance to where $\theta/\theta_w = 0{\cdot}1$. The maximum thickness of the thermal boundary layer is at $x_1 = 2\delta = 2^{\frac{1}{2}}$ and from (19) is found to be

$$y_{1\delta} = 2{\cdot}79/u^{\frac{1}{2}}. \tag{21}$$

E. Thermal convective plumes

We next solve for the temperature distribution in the convective plumes. The centre-line of each plume is in fact the division between the adjacent convective cells. However, it is convenient to determine the structure of the entire plume which in fact belongs to two adjacent cells. Since the plumes are thin, it is appropriate to take v to be independent of x within the plumes. As in the horizontal boundary layer, we will assume that v is not a function of y and take the mean value given in (16). Therefore we take v = constant and $u = 0$ in the solution for the structure of the plumes. The analysis is valid for both the hot and cold plumes.

The governing energy equation is the same as for the thermal boundary layers except that x and y are interchanged,

$$v(\partial\theta/\partial y_2) = \partial^2\theta/\partial x_2^2, \tag{22}$$

where y_2 is the distance from the horizontal surface where the plume is formed and x_2 is the distance from the centre-line of the plume. However, the boundary conditions for the plume structure differ from those for the boundary-layer structure. It is required that $\theta \to 0$ as $x_2 \to \pm\infty$ and the initial temperature profile, at $y_2 = 0$, is specified. The solution of the heat equation with these boundary conditions is known as Laplace's solution and is given by (see Carslaw & Jaeger 1959)

$$\theta = \frac{1}{2}\left(\frac{v}{\pi y_2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \theta_0 \exp\left[-\frac{(x_2 - x')^2 v}{4y_2}\right] dx', \tag{23}$$

where θ_0 is the initial temperature distribution at $y_2 = 0$.

We assume that the initial temperature distribution in the plume is the same as the temperature distribution in the thermal boundary layers adjacent to the base of the plume at $x_1 = 2\delta = 2^{\frac{1}{2}}$ with the result

$$\theta_0 = \frac{1}{2}\left[1 - \frac{1}{\pi^{\frac{1}{2}}} \int_0^{x' u^{\frac{1}{2}}/2^{\frac{1}{4}}} e^{-z^2/4} dz\right]. \tag{24}$$

Substitution of (24) into (23) and changing the variable of integration gives

$$\theta = \frac{1}{2}\left(\frac{1}{\pi y_2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left[1 - \frac{2}{\pi^{\frac{1}{2}}} \int_0^{(u/v)^{\frac{1}{2}}|\xi|/2^{\frac{1}{4}}} e^{-\eta^2} d\eta\right] \exp\left[-\frac{(\frac{1}{2}x_2 v^{\frac{1}{2}} - \xi)^2}{y_2}\right] d\xi. \tag{25}$$

From (25) the temperature distribution in each of the convective plumes can be determined.

F. Stagnation point thermal boundary layers

Where the convective plumes impinge on the horizontal surface stagnation point thermal boundary layers are formed. Adjacent to each stagnation point the velocity components are given by $u = 2Ax_3$ and $v = -2Ay_3$ where x_3 and y_3 are measured from the stagnation point. From figures 2 and 3 it is found that $A \doteq 0{\cdot}75\gamma$. Substitution of these velocity components into the energy equation gives

$$1{\cdot}5\gamma[x_3(\partial\theta/\partial x_3) - y_3(\partial\theta/\partial y_3)] = (\partial^2\theta/\partial x_3^2) + (\partial^2\theta/\partial y_3^2). \tag{26}$$

We will assume that the temperature outside the stagnation point thermal boundary layers is the core temperature $\theta = 0$. The actual temperature is that given by the plume solution. The boundary conditions that we require are $\theta = 0$ as $y_3 \to \infty$ and $\theta = \frac{1}{2}$ at $y_3 = 0$. A similarity solution that satisfies these boundary conditions is obtained by setting $\theta = \theta(y_3)$ with the result

$$-1{\cdot}5\gamma y_3(d\theta/dy_3) = d^2\theta/dy_3^2. \tag{27}$$

The solution of this equation is

$$\theta = \frac{1}{2}\left[1 - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{y_3(1{\cdot}5\gamma)^{\frac{1}{2}}} e^{-z^2/2} dz\right]. \tag{28}$$

The corresponding value for the local Nusselt number valid in the vicinity of the stagnation point is

$$Nu_l = (0{\cdot}75\gamma/\pi)^{\frac{1}{2}}. \tag{29}$$

If we define the thickness of the stagnation point thermal boundary layer, $y_{3\delta}$, to be the distance to where $\theta/\theta_w = 0{\cdot}1$, we obtain

$$y_{3\delta} = 1{\cdot}64/(1{\cdot}5\gamma)^{\frac{1}{2}}. \tag{30}$$

G. Matching of solutions

It is now necessary to evaluate the constant γ which remains an unknown in the core solution and determines the magnitude of the convective velocities. In order to obtain the appropriate boundary condition on $\partial v/\partial x$ at $x = \pm\delta$ we write the y-component of the momentum equation valid within the convective plume, from (10),

$$(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) = (\partial P/\partial y) - R\theta. \tag{31}$$

Within the thin convective plume it is consistent to neglect both $\partial^2 v/\partial y^2$ and $\partial P/\partial y$ compared with $R\theta$. Integrating the momentum equation noting that $\partial v/\partial x = 0$ on the centre-line of the plume, we obtain

$$\gamma \equiv -\frac{\partial v}{\partial x_2}\bigg|_{\delta} = R\int_0^{\infty} \theta \, dx_2, \tag{32}$$

and γ is proportional to the integral of the temperature deficit (excess) in the plume. Since there is no heat addition to the plume γ is independent of y_2 and is independent of the plume structure because the initial temperature distribution at $y_2 = 0$ can be used to evaluate the integral in (32). Substitution of (24) into (32) and integrating gives

$$\gamma = 0{\cdot}671(R/u^{\frac{1}{2}}). \tag{33}$$

Substituting (33) into (17) and solving for u gives the mean dimensionless velocity on the horizontal boundaries

$$u = 0{\cdot}142R^{\frac{2}{3}}. \tag{34}$$

Substitution of (34) into (33) gives

$$\gamma = 1{\cdot}78R^{\frac{2}{3}}, \tag{35}$$

and substitution of (35) into (16) gives the mean dimensionless vertical velocity on the boundaries between cells

$$v = 0{\cdot}250R^{\frac{2}{3}}. \tag{36}$$

Having obtained the dimensionless velocities, it is now possible to determine the local heat flux to the boundaries expressed in terms of a local Nusselt number. The local Nusselt number of the thermal boundary layers is obtained by substituting (34) into (20)

$$Nu_l = 0{\cdot}106(R^{\frac{1}{3}}/x_1^{\frac{1}{2}}). \tag{37}$$

The local Nusselt number for the stagnation point thermal boundary layer is obtained by substituting (35) into (29)

$$Nu_l = 0{\cdot}652R^{\frac{1}{3}}. \tag{38}$$

To a first approximation it is appropriate to assume that (38) replaces (37) when the Nusselt number obtained from (38) is smaller than the Nusselt number obtained from (37), that is for $x_1 < 0{\cdot}0264$. The Nusselt number for the total heat transfer between the horizontal surfaces is obtained by taking the mean value of the local Nusselt number over the cell. Using the values of the local Nusselt number given above, the total Nusselt number is

$$Nu = \frac{1}{2^{\frac{1}{2}}}\int_0^{2^{\frac{1}{2}}} Nu_l\, dx_1 = 0{\cdot}167 R^{\frac{1}{3}}. \tag{39}$$

The maximum thickness of the thermal boundary layer is obtained by substituting (34) into (21)

$$y_{1\delta} = 7{\cdot}38/R^{\frac{1}{3}}. \tag{40}$$

Since $y_{1\delta}$ is the ratio of the actual thickness to the distance between the horizontal boundaries, $y_{1\delta}$ must be small compared with one for the boundary layer hypothesis to be valid. From (40) we see that the boundary layers are thin for sufficiently large values of the Rayleigh number. Since the thickness of the convective plumes are of the same order as the thickness of the boundary layers, the plumes will also be thin for large values of the Rayleigh number. Substitution of (35) into (30) gives the non-dimensional thickness of the stagnation point thermal boundary layer

$$y_{3\delta} = 1{\cdot}00/R^{\frac{1}{3}}. \tag{41}$$

3. Application to mantle convection

Before the theory developed for cellular convection can be applied to mantle convection we must obtain values for such properties as the thermal diffusivity, the kinematic viscosity, and the coefficient of volume expansion. The distance d over which mantle convection occurs must be prescribed and the temperatures on the boundaries must be given. Some of these quantities are not well known and some vary in an unknown manner. For these reasons close agreement between observed phenomena and theoretical predictions must be regarded as somewhat fortuitous.

Probably the most serious uncertainty relates to the kinematic viscosity ν. This problem is discussed by McConnell (1965) and following him and others a value of 10^{22} cm^2/sec is chosen; this, however, is subject to an uncertainty of one or two orders of magnitude. In comparison, the uncertainties in the other variables are less important. Following Verhoogen (1958), Knopoff (1964), Jaeger (1965) and Tozer (1965), the following values are assigned: coefficient of thermal expansion, α, 2×10^{-5} °K^{-1}; average thermal gradient in excess of the adiabatic gradient, β, $1{\cdot}5 \times 10^{-5}$ °K/cm; thermal diffusivity, κ, 10^{-2} cm^2/sec. The acceleration due to gravity remains approximately constant with depth, $g = 10^3$ cm/sec^2, and the depth of the convecting layer, d, is taken to be $1{\cdot}5 \times 10^8$ cm corresponding to a cell width, $2\delta d$, of 2100 km. These values give a Rayleigh number $R = 1{\cdot}5 \times 10^6$. This value of the Rayleigh number is uncertain by at least one order of magnitude. Virtually all published values for the Rayleigh number valid for the earth's mantle fall in the range 10^5 to 10^8.

Using $R = 1{\cdot}5 \times 10^6$ and the values of κ and d given above, the mean horizontal velocity on the surface is found from (34) to be $1{\cdot}24 \times 10^{-7}$ cm/sec. This value is in excellent agreement with the velocities of continental drift obtained from paleomagnetic and other evidence; e.g. 10^{-7} cm/sec Tozer (1965), 1·25 cm/yr Orowan (1965), 3–5 cm/yr Allen (1965). It should be emphasized that such excellent agreement must be considered fortuitous.

From (37) and (38) the local Nusselt number can be determined for the heat flux to the surface above the ascending limb and as a function of the distance

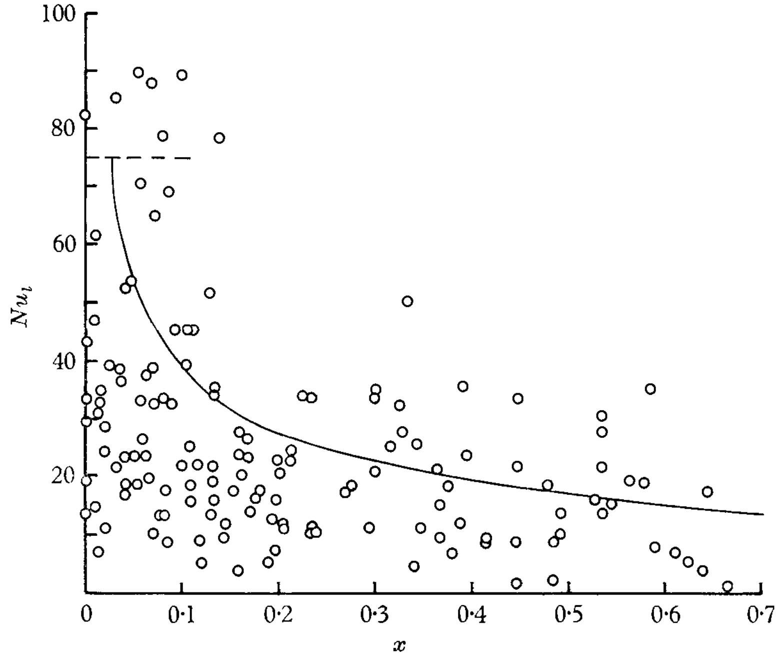

Figure 4. Comparison between the measured values of Nusselt number at various dimensionless distances from the East Pacific Rise and theory for a Rayleigh number of $1{\cdot}5 \times 10^6$. O, Lee & Uyeda (1965); ——, equation (37); – – –, equation (38).

from the ascending limb. The predicted heat flux can be compared with measurements made on the ocean floors. In figure 4 the measured values of the Nusselt number are plotted against the dimensionless distance from the crest of the East Pacific Rise. The experimental points are taken directly from a figure published by Lee & Uyeda (1965). The Nusselt numbers are determined from the measured heat fluxes using the values of parameters given above and a thermal conductivity, k, of 6×10^{-3} cal/cm sec °K (Clark & Ringwood 1964; Jaeger 1965; McDonald 1965). Also included in figure 4 are the Nusselt numbers given by (37) and (38) for a Rayleigh number of $1{\cdot}5 \times 10^6$. It is seen that agreement between theory and experiment is excellent considering the scatter in the measurements. Since the heat transfer is dependent on the one-third power of the Rayleigh number, the theoretical predictions should be accurate to less than an order of magnitude.

Although high values of q, and thus Nu_l, are measured only in the central parts of the Rise, low values are measured everywhere and there is a resultant wide scatter of points. This is caused in part by the plotting together of data from different areas of the ridge system. However, the principal reason for the scatter may be due to the mechanism of heat transfer in the axial zones of the ridges. As crystalline silicate compounds are convected upwards in the ascending limb of a convective cell, partial melting through pressure release is expected to occur at a depth of about 100 km. As a result, molten silicate rises to the surface where it is extruded as submarine, basaltic lava up to distances of 300 km from the ridge crest. Low values of heat flux in the axial zone have probably been measured in areas where there has been recent rapid deposition of debris after submarine volcanic explosions.

4. Discussion and conclusions

It would be desirable to correlate the theory given in this paper with laboratory experiments. Silveston (1958) has measured the total heat transfer between two surfaces at a Prandtl number of 3000 and Rayleigh numbers up to 30,000. However, these experiments were carried out between parallel disks so that the free-surface boundary conditions used in the present analysis are invalid. At large Prandtl and Rayleigh numbers Silveston observed irregular elongated rolls which resemble the cell structure associated with mantle convection.

The question of transition to turbulent convection should also be considered. With the large Prandtl numbers for mantle convection, any kind of viscous transition to turbulence will certainly be inhibited. The laboratory experiments have not been carried out at sufficiently high Prandtl and Rayleigh numbers to verify this suppression of turbulence. The elongated rolls associated with mantle convection would indicate that the flow is laminar.

The agreement between the theory presented in this paper and measurements associated with mantle convection is certainly satisfactory. The free-surface boundary condition used in the analysis is valid for the upper surface in mantle convection. However, the appropriate boundary condition for the lower surface in mantle convection is far less certain. It is expected that the properties of the mantle such as viscosity may vary considerably with depth. Since the appropriate values are not well known, it is questionable whether a theory taking into account the variation of the properties of the mantle is required.

It seems reasonable to conclude that the quantitative agreement between theory and measurement given in this paper strongly favours the hypothesis of mantle convection. The quantitative distribution of velocities and temperatures which the theory provides should help the understanding of processes both on and beneath the earth's surface.

This work was carried out while one of the authors (D. L. T.) held a National Science Foundation Postdoctoral Research Fellowship.

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